ON JACKSON'S THEOREM

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Abstract

We prove that for a function \( f \in W_p^1[-1,1] \), \( 0 < p < 1 \) and \( n, r \) in \( \mathbb{N} \) (the set of natural numbers), we have

\[
\int_{-1}^{1} f(x)dx - \sum_{j=1}^{n} \omega_j f(x_j) \leq c(r)n^{-1} \int_{0}^{1} \omega_{\psi^{-1}}(f', u) \frac{du}{u^2}
\]

where \(-1 < x_1 < \ldots < x_n < 1\) are the roots of Legendre polynomial, and \( \omega_{\varphi}^m(g, \delta) \), is the Ditzian-Totik mth modulus of smoothness of \( g \) in \( L_p \).

1. Introduction

Let \( L_p, 0 < p < \infty \) be the set of all functions, which are measurable on \([a, b]\), such that

\[
\|f\|_{L_p[a,b]} := \left( \int_{a}^{b} |f(x)|^p dx \right)^{1/p} < \infty.
\]

And let \( W_p^r[a, b] \) be the space of functions that \( f^{(r)} \in L_p[a, b] \) and \( f^{(r-1)} \) is absolutely continuous in \([a, b]\).

We believe that for approximation in \( L_p, p < 1 \) the measure of smoothness \( \omega_{\varphi}^r(f, \delta) \) introduced by Ditzian and Totik [1] is the appropriate tool. Recall that

\[
\omega_{\varphi}^r(f, \delta, [a, b])_p = \sup_{0 < h < \delta} \left( \int_{a}^{b} |\Delta_{h}\varphi(f, x, [a, b])|^p dx \right)^{1/p},
\]

where
\[ \Delta^r_{h\phi}(f, x, [a, b]) := \begin{cases} \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f \left( x - \frac{rh}{2} + kh \right), & \text{if } x \pm \frac{rh}{2} \in [a, b] \\ 0, & \text{o.w.} \end{cases} \]

For \([a, b] := [-1, 1]\) for simplicity we write \(\|\|_p = \|\|_{L_p[a, b]}\), and \(\omega^r_{\phi}(f, \delta)_p := \omega^r_{\phi}(f, \delta, [a, b])_p\).

Recall that the rate of best nth degree polynomial approximation is given by
\[ E_n(f)_p := \inf_{p_n \in \Pi_n} \|f - p_n\|_p \]
where \(\Pi_n\) denote the set of all algebraic polynomials of degree not exceeding \(n\).

To prove our theorem we need the following direct result given by:

**Theorem 1.1.** [2] For \(n, r\) in \(N\) and \(f \in L_p[-1, 1]\)
\[ E_n(f)_p \leq c \omega^r_{\phi}(f, n^{-1})_p \]
where \(c\) is a constant depending on \(r\) and \(p\) (if \(p < 1\)). For \(1 \leq p \leq \infty\) (1) was proved by Ditzian and Totik [1] and for \(0 < p < 1\), it has been proved by DeVore, Leviatan and Yu [2].

Now, consider the Gaussian Quadrature process [3]
\[ \frac{1}{n} \int_{-1}^{1} f(x)dx \approx \sum_{j=1}^{n} \omega_j f(x_j) =: I_n(f) \]
(2)
based on the roots \(-1 < x_1 < \ldots < x_n < 1\) of the \(n\)th Legendre polynomial. Since this exact polynomial of degree less than \(2n\), we get for the error
\[ e_n(f) = \frac{1}{n} \int_{-1}^{1} f(x)dx - I_n(f) \]
in (2) by the definition of the degree of best approximation we have
\[ e_n(f) \leq 2E_{2n-1}(f)_\infty \]
where
\[ \|f\|_{\infty} := \sup_{x \in [-1, 1]} |f(x)| \]
(note that $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j$). The crude method of estimating $e_n(f)$ consists of applying Jackson estimate on the right of (3) from (1) we get the sharp inequality

$$e_n(f) \leq c_\omega \varphi^r(f, n^{-1})_\infty$$

which already takes in to account the possibly less smooth behavior of $f$ at $\pm 1$. However the supremum norm in (5) is still too rough, and the natural question is whether for smooth functions one can get upper bounds for $e_n(f)$ using certain $L_p, p < 1$ quasi-norm.

R. A. DeVore and L. R. Scott [3] found such estimates, they proved

$$e_n(f) \leq c(s)n^{-s} \int_{-1}^{1} \left| f^{(s)}(x) \right| \left( 1 - x^2 \right)^{s/2} dx$$

(5)

first for $s=1$ which obviously implies

$$e_n(f) \leq cn^{-1} E_{2n-2}(f', \varphi) \quad p \geq 1$$

(6)

where $E_n(f)_{\varphi, p}$ means the best weighted approximation with weight $\varphi(x)$ of $f$ in $L_p$ defined by

$$E_n(f)_{\varphi, p} := \inf_{p_n \in \Pi_n} \| \varphi(f - p_n) \|_p.$$ 

They then proceeded to estimate $E_n(f')_{p}, p \geq 1$, using higher derivatives of $f$ which finally yielded (5) for any $s \geq 1$.

2. The main result

In this section we introduce our main result. Using (6) we obtain the following theorem

**Theorem 2.1.** For $f \in W^{1,1}_p[-1,1], 0 < p < 1$ we have

$$e_n(f) \leq c(r)n^{-1/n} \varphi^{-1}(f', u)^{p} du$$

(7)

Of course the convergence of the integral on the right implies that $f$ is $L_p$ equivalent of a locally absolutely continuous function. We use this equivalent
representative of \( f \) in the quadrature formula (Otherwise, we don’t have even \( e_n(f) = o(1) \))

**Proof.** Let \( p_n \in \Pi_n \) be the best approximating polynomial for \( f \) in 
\[
L_p[-1,1], \quad p < 1.
\]
Then 
\[
f = p_n + \sum_{k=0}^{\infty} \left( p_{2^{k+1} n} - p_{2^k n} \right)
\]
in \( L_p[-1,1] \) (i.e. the expression in the right is the \( L_p \) equivalent of \( f \) which we need). From (6) and Markov-Bernstein type inequality (see for example [4])
\[
e_n(f) \leq c n^{-1} E_{2n-2}(f') q, \quad q \geq 1
\]

\[
\leq c n^{-1} E_n(f') q, \varphi
\]

\[
\leq c n^{-1} \| \varphi(f' - p_n') \|_q
\]

\[
\leq c n^{-1} \sum_{k=0}^{\infty} 2^{k+1} n \| \varphi(p_{2^{k+1} n} - p_{2^k n}') \|_q.
\]

Then using the fact that any two quasi norms are equivalent on the space of algebraic polynomials of a fixed degree we have
\[
e_n(f) \leq c(p) \sum_{k=0}^{\infty} 2^{k+1} n E_{2^k n}(f) p, \quad p < 1.
\]

In view of (1) we get 
\[
e_n(f) \leq c(p) \sum_{k=0}^{\infty} 2^k n \omega_{\varphi}^r(f,2^{-k} n^{-1}) p.
\]

Now since \( f \in W_p^1[-1,1], 0 < p < 1 \), so that 
\[
e_n(f) \leq c(p)n^{-1} \sum_{k=0}^{\infty} 2^k n \omega_{\varphi}^r(f,2^{-k} n^{-1}) p
\]

\[
\leq c(p)n^{-1/2} \omega_{\varphi}^{r-1}(f',u) p du.
\]

Provided the last integral convergence ♦

As a final remark, we mention that similar bounds holds for many other systems of nodes and in (7) the right hand side has the order 
\[
\left( \int \frac{\| f \|^p}{x_n} + \int \| f \|^p \right)^{1/p},
\]
for any \( f \) constructed from analytic functions, \( |x| \) and iterated logarithms of these, which means that (7) is the best possible estimate for such functions.
References


