On Semi Feebly Open Set and its Properties

1. Introduction

The topological idea from study this set is generalization the properties and using it's to prove the theorems. In [7] N. Leven (1963) gives the definition of semi open (s-open) set, semi closed (s-closed) set and studies the properties of it. He defined a set A named (s-open) set in topological space $\chi$ if find an open set $O \subseteq A \subseteq \overline{O}$ where $\overline{O}$ denoted by the closure of $O$ in $\chi$, the complement semi-open (s-open) set called semi-closed (s-closed) set. In (1971) S. G. Crossiey and S. K. Hildebrand defined the concept semi closure and they defined it, the semi closure of a set $A$ in topological space $\chi$ is the smallest semi-closed (s-closed) set contained $A$ [2] and shortened by $\text{Scl}(A)$ or $\overline{A}^s$. The truth $\overline{A}^s$ is the intersection of all semi closed sets contained $A, \overline{A}^s \subseteq \overline{A} \text{ and } \overline{A}^s = \overline{A}^s$. Maheswari and Tapi (1978) in [3] defined feebly closed (f-closed), feebly open (f-open) set. A set $A$ in a topological space $\chi$ named feebly open (f-open) set in $\chi$ if find an open set $V$ such that $V \subseteq A \subseteq \overline{V}$. A set $A$ in a topological space $\chi$ is feebly closed
if it is complement is feebly open. Every open is \((f\text{-open})\)set, but the converse may be not true. Every closed is \((f\text{-closed})\) set, but the converse may be not true.

We will use the\( tp\)-symbol to denote the topological space, \((s\text{-open})\) to semi open set,\( (s\text{-closed})\) to semi closed set,\( (f\text{-open})\) to feebly open set and \((f\text{-closed})\) to feebly closed set. wherever it is found in this paper.

2. Preliminaries

**Definition(2.1)**[7]: Assume that \((X, t)\) is a \(tp\)-\& \(A \subseteq X\). Then \(A\) is named \(s\text{-open}\) in \(X\) if there exists \(O \in t: O \subseteq A \subseteq \overline{O}\). Or equivalent [5], \(A\) called \(s\text{-open}\) in \(X\) \iff \(A \subseteq \overline{A}^s\), equivalent \(\overline{A} = \overline{A}^s\), the complement of \(s\text{-open}\) is named \(s\text{-closed}\).

**Definition(2.2)**[7]: Let \((X, t)\) be \(tp\)-\& \(A \subseteq X\) then \(A\) called \(s\text{-closed}\) in \(X\) if there exists a closed set \(F\) such that \(F^s \subseteq A \subseteq F\), or equivalent[5], \(A\) is \(s\text{-closed}\) in \(X\) \iff \(\overline{A}^s \subseteq A\), equivalent \(A^s = \overline{A}^s\).

**Definition(2.3)**[5]: Let \((X, t)\) be \(tp\)-\& \(A \subseteq X\), then the intersection of all \(s\text{-closed}\) subset of \(X\) contained \(A\) named \((s\text{-closure})\) of \(A\) and the union of all \(s\text{-open}\) subset of \(X\) contained \(A\) named \((s\text{-interior})\) of \(A\) and are shortened by \(\overline{A}^s\), \(A^s\) respectively.

**Proposition (2.4)**[7]: Let \(\{A_\lambda\}_{\lambda \in \Lambda}\) be a family of \(s\text{-open}\) in a \(tp\)-\(X\) then \(\bigcup_{\lambda \in \Lambda} A_\lambda\) is \(s\text{-open}\).

**Proposition (2.5)**[7]: Let \(X\) be a \(tp\)-\(X\) then the intersection of two \(s\text{-open}\) in \(X\) does not need to be \(s\text{-open}\).

**Example (2.6)**: Let \(X = \{k, v, h\}, t = \{\{k\}, \{v\}, \{k, v\}, X, \emptyset\}\) then each of \(\{k, h\}, \{v, h\}\) are \(s\text{-open}\), but \(\{k, h\} \cap \{v, h\} = \{h\}\) not \(s\text{-open}\).

**Definition(2.7)**[4]: The intersection of every semi open subset of \(tp\)-\(X\) contained a set \(A\) is named Semi kernel of \(A\) and shortened by \((S\text{ker}(A))\).

Means that: \(S\text{ker}(A) = \cap \{U : U\ \text{s-open and } A \subseteq U\}\).

**Definition(2.8)**[8]: A set \(A\) in a \(tp\)-\(X\) called \(f\text{-open}\) in \(X\) if there exists an open set \(V\) such that \(V \subseteq A \subseteq \overline{V}^s\), or equivalent, a set \(A\) called \(f\text{-open}\) in \(X\) if and only if \(A \subseteq \overline{A}^s\), the complement of \(f\text{-open}\) is called \(f\text{-closed}\) that \(\overline{A}^s \subseteq A\).

**Remark(2.9)**[6]: Let \((X, t)\) be \(tp\)-\& \(A \subseteq t\) then \(A\) is \(f\text{-open}\) and \(A^c\) is \(f\text{-closed}\).

But the converse is not true in general as in the next example.

**Example (2.10)**: Assume that \(X = \{1, 2, 3, 4, 5\}\) and \(t = \{\emptyset, X, \{1\}, \{2, 4\}, \{1, 2, 4\}\}\) then, \(f\text{-open}\) sets are \(\{\emptyset, X, \{1\}, \{2, 4\}, \{1, 2, 4\}\}\), \(f\text{-closed}\) sets are \(\{\emptyset, X, \{2, 3\}, \{1, 3, 5\}, \{3, 5\}, \{5\}\}\).

Take \(A = \{1, 2, 3, 4\}\) is \(f\text{-open}\), but it not open set & \(A^c = \{5\}\) \(f\text{-closed}\), but it is not closed.

**Proposition (2.11)**[9]: Assume that \((X, t)\) is a \(tp\)-\(A, B\) subsets of \(X\) then:
1. $\overline{A}^f$ $f$-closed.

2. $A \subseteq \overline{A}^f$.

3. $A$ is ($f$-closed) $\iff A = \overline{A}^f$.

4. $A \subseteq B \implies \overline{A}^f \subseteq \overline{B}^f$.

5. If $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of subset of $\chi$ then $\bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f = \bigcup_{\lambda \in \Lambda} \overline{A_\lambda}^f$.

6. If $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of subset of $\chi$ then $\bigcap_{\lambda \in \Lambda} \overline{A_\lambda}^f \subseteq \bigcap_{\lambda \in \Lambda} \overline{A_\lambda}^f$.

7. $\overline{\overline{A}}^f = \overline{A}$.

8. $\overline{A}^f \subseteq \overline{A}$.

9. $\overline{\overline{A}}^f = \overline{A}^f = \overline{A}$.

10. $\overline{A}^f = A \cup A'^f$.

11. $\overline{A}^f = A \cup \overline{A}$.

12. $x \in \overline{A}^f \iff$ any $f$-open $G$ contained $x$, $A \cap G \neq \emptyset$.

**Proposition (2.12) [9]:** Let $\chi$ be a $tp$-s & $A, B$ subset of $\chi$ where $B$ $f$-open, If $x \in B$ and $A \cap B = \emptyset$ then $x \notin \overline{A}^f$.

**Definition (2.13) [10]:** Let $\chi$ be a $tp$-s a subset $A$ of $\chi$ is said to be

i. Dense or (every dense) in $\chi \iff \overline{A} = \chi$.

ii. Nowhere dense or (non-dense) in $\chi$ iff $(\overline{A})^s = \emptyset$.

**Definition (2.14) [5]:** Let $(\chi, t)$ be a $tp$-s and $A \subseteq \chi$, $A$ is named preopen ($p$-open) if $A \subseteq \overline{A}$ and $A^c$ is named pre closed ($p$-closed) that $\overline{A^c} \subseteq A$.

**Lemma (2.15) [4]:** Every singleton $\{x\}$ in a $tp$-s $\chi$ is either nowhere dense or preopen.

3. The Main Results

**Definition (3.1):** Assume that $(\chi, t)$ is a $tp$-s then a subset $A$ in a space $\chi$ is named semi feebly open ($sf$-open) set in a space $\chi$ if $A \subseteq U$ where $U$ semi open set in $\chi$ then $\overline{A}^f \subseteq U$. The complement of semi feebly open is called semi feebly closed ($sf$-closed) it is as follows $U \subseteq \overline{A}^f$ where $U$ semi closed set in $\chi$. 
**Example (3.2):** Let $X = \{k, v, h\}$, $\tau = \{X, \emptyset, \{k\}\}$ then

open set: $\{X, \emptyset, \{k\}\}$, closed set: $\{\emptyset, X, \{v, h\}\}$

$s$-open: $\{\emptyset, X, \{k\}, \{k,v\}, \{k,h\}\}$, $s$-closed: $\{\emptyset, X, \{v, h\}, \{h\}, \{v\}\}$

$f$-open: $\{\emptyset, X, \{k\}, \{k,v\}, \{k,h\}\}$, $f$-closed: $\{\emptyset, X, \{v, h\}, \{h\}, \{v\}\}$

$sf$-open $\Rightarrow \{\emptyset, X, \{v\}, \{h\}, \{v, h\}\}$

we notes that $\{\{k\}, \{k,v\}, \{k,h\}\}$ not $sf$-open because $\{k\} \subseteq \{k\}$ where $\{k\}$ $s$-open, but $\{k\}^f = X \not\subseteq \{k\}$, $\{k\}$ is not $sf$-open.

$\{k, v\} \subseteq \{k, v\}$ where $\{k, v\}$ $s$-open, but $\{k, v\}^f = X \not\subseteq \{k, v\}$, $\{k, v\}$ is not $sf$-open.

$\{k, h\} \subseteq \{k, h\}$ where $\{k, h\}$ $s$-open, but $\{k, h\}^f = X \not\subseteq \{k, h\}$, $\{k, h\}$ is not $sf$-open.

**Remark (3.3):** Each $f$-closed is $sf$-open.

Proof: Let $A$ be $f$-closed set in a tp-$sX$. $A \subseteq U$, $U$ $s$-open, $A$ is ($f$-closed) set then $A = \overline{A}^f$ and $A \subseteq \overline{A}^f \subseteq U \Rightarrow A$ is ($sf$-open) set.

The converse of (Remark (3.3)) is not true in general, as in the next example shows:

**Example (3.4):** Let $X = \{1,2,3,4,5\}$, $\tau = \{X, \emptyset, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5\}\}$ and $A = \{1,2,4,5\}$ then $A$ is $sf$-open not $f$-closed.

Proof: The open sets are $\{X, \emptyset, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5\}\}$,

the closed sets are $\{\emptyset, X, \{2,3,4,5\}, \{1,2,5\}, \{2,5\}, \{1\}\}$ and

$s$-open sets are $\{X, \emptyset, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5\}, \{2,3,4\}, \{3,4\}, \{1,2,3,4\}, \{1,3,4,5\}\}$

Let $A = \{1,2,4,5\}$ and $U = X$ which is $s$-open, then $A \subseteq U$

$\overline{A}^f = \{1,2,4,5\} \cup \{1,2,4,5\}^\circ$ [Proposition (2.11)(11)]

$\overline{A}^f = \{1,2,4,5\} \cup X = X \subseteq U = X$ $\Rightarrow A$ $sf$-open set, but $A$ not $f$-closed because $\{1,2,4,5\}^\circ = X \not\subseteq \{1,2,4,5\}$.

**Remark (3.5):** Every closed set is $sf$-open set.

Proof: Let $A$ closed set $\Rightarrow A$ is $f$-closed by [Remark (1.15)] $A$ is $sf$-open set.

But the converse of (Remark (3.5)) in general is not true as in our [Example (3.4)] shows:

$A = \{1,2,4,5\}$ $sf$-open, $\overline{A} = X \neq A \Rightarrow \overline{A} \neq A$ then $A$ is not closed.
The next diagram explains the relationship these types of sets.

![Diagram showing relationships between types of sets]

**Notes (3.6)**

For each \( tp-s \) \( \emptyset \), \( X \) are \( sf \)-open.

Every subset of discreet or indiscreet \( tp-s \) is \( sf \)-open.

Every closed interval in \((R, U)\) where \( U \) is usual topology is \( sf \)-open.

**Proposition (3.7):** Let \( X \) be \( tp-s \) then the union of all \( sf \)-open sets in \( X \) is also \( sf \)-open set.

Proof: Let \( \bigcup_{\lambda \in \Lambda} A_{\lambda} \subseteq U \), \( U \) semi-open in a topological space \( X \) then

\[
A_{\lambda} \subseteq \bigcup_{\lambda \in \Lambda} A_{\lambda} \subseteq U \Rightarrow A_{\lambda} \subseteq U \Rightarrow \overline{A_{\lambda}}^{f} \subseteq U \Rightarrow \bigcup_{\lambda \in \Lambda} \overline{A_{\lambda}}^{f} \subseteq U
\]

since \( \{A_{\lambda}\}_{\lambda \in \Lambda} \) be a collection of all subset of \( X \) then \( \overline{\bigcup_{\lambda \in \Lambda} A_{\lambda}}^{f} = \bigcup_{\lambda \in \Lambda} \overline{A_{\lambda}}^{f} \)

\[
\Rightarrow \bigcup_{\lambda \in \Lambda} \overline{A_{\lambda}}^{f} = \bigcup_{\lambda \in \Lambda} \overline{A_{\lambda}}^{f} \subseteq U \Rightarrow \overline{\bigcup_{\lambda \in \Lambda} A_{\lambda}}^{f} \subseteq U \text{ is } \bigcup_{\lambda \in \Lambda} A_{\lambda} \text{ sf-Open.}
\]

**Proposition (3.8):** Let \( A \) be nowhere dense in a \( tp-s \) \( X \) then \( A \) is \( f \)-closed.
Proof: Assume that $A$ is a subset of $\mathcal{X}$ so that $A$ is nowhere dense then $(\overline{A})^* = \emptyset$ and $(\overline{A})^\circ = \emptyset$, but $\emptyset \subseteq A \Rightarrow A$ is $f$-closed set.

**Lemma (3.9):** A subset $A$ of a tp-$s$ $\mathcal{X}$ is sf-open iff $\overline{A}^f \subseteq \text{Sker} (A)$.

Proof: ($\Rightarrow$) Let $A$ be sf-open in $\mathcal{X}$, then $\overline{A}^f \subseteq U$ when $\emptyset \subseteq U$ and $U$ is sf-open in $\mathcal{X}$, this emplace $\overline{A}^f \subseteq \cap \{U : A \subseteq U \text{ and } U \in \text{s-open}(X)\} = \text{S ker} (A)$. ($\Leftarrow$) Conversely, assume that $\overline{A}^f \subseteq \text{S ker} (A) \Rightarrow \overline{A}^f \subseteq \cap \{U : A \subseteq U \text{ and } U \in \text{s-open}(X)\} \Rightarrow \overline{A}^f \subseteq U$ for all s-open set $U$ in $\mathcal{X}$.

**Proposition (3.10):** Let $\mathcal{X}$ be tp-$s$, then the arbitrary intersection of sf-open sets in $\mathcal{X}$ is sf-open set.

Proof: Let $\{A_\lambda : \lambda \in \Lambda \}$ be arbitrary collection of sf-open sets in a space $\mathcal{X}$, and let $A = \cap A_\lambda, \lambda \in \Lambda$, let $x \in \overline{A}^f$ then by [Lemma(2.15)] we consider the following two cases.

**Case 1:** $\{x\}$ is nowhere dense

If $x \notin A$ then for some $\alpha \in \Lambda$ we have $x \notin A$, as nowhere dense subset are feebly closed [Proposition(3.8)] there fore $x \notin \text{S ker} (A)$. On the other hand by [Lemma(3.9)] $A_\alpha$ is sf-open, then, $x \in \overline{A}^f \subseteq \overline{A}_{\alpha}^f \subseteq \text{S ker} (A)$ paradoxically $x \in A$ and hence $x \in \text{S ker} (A) \Rightarrow \overline{A}^f \subseteq \text{S ker} (A)$, $A$ sf-open set.

**Case 2:** $\{x\}$ is preopen, Let $F = \overline{\{x\}}^\circ$ and $x \notin \text{S ker} (A)$, $\exists$ semi closed set $C$ containing $X$, so that $C \cap A = \emptyset$, $x \in F = \overline{\{x\}}^\circ \subseteq \overline{C}^\circ \subseteq C$. As $F$ is an open set containing $x$ and $x \in \overline{A}^f$ therefore, $F \cap A \neq \emptyset$ as $F \subseteq C \Rightarrow C \cap A = \emptyset$ paradoxically $x \in \text{S ker} (A) \Rightarrow A$ sf-open set.

**Proposition (3.11):** If $A$ is sf-open and $B$ f-closed in a tp-$s$ $\mathcal{X}$ then $A \cap B$ is sf-open in $\mathcal{X}$.

Proof: Assume that $A \cap B \subseteq U$ where $U$ is s-open set then $A \cap B \cap U^c = \emptyset$

$\Rightarrow A \cap (B \cap U^c) = \emptyset \Rightarrow A \subseteq B^c \cup U$, but $B^c \cup U$ s-open $\overline{A}^f \subseteq B^c \cup U$

$\Rightarrow \overline{A}^f \cap (B^c \cup U)^c = \emptyset \Rightarrow \overline{A}^f \cap B \subseteq U \Rightarrow \overline{A} \cap \overline{B}^f \subseteq U \Rightarrow A \cap B$ sf-open.

**Proposition (3.12):** Assume that $\mathcal{X}$ is a tp-$s$ $\& A \subseteq X$ then $\overline{A}^f$ is sf-open set.

Proof: Let $\overline{A}^f \subseteq G$ where $G$ is s-open set, since $\overline{\overline{A}^f}^f = \overline{A}^f \subseteq U$

$\Rightarrow \overline{\overline{A}^f}^f \subseteq G$ \Rightarrow $\overline{A}^f$ sf-open set.

**Proposition (3.13):** Assume that $\mathcal{X}$ is a tp-$s$ $\& A \subseteq X$ then $\overline{A}$ is sf-open set.
Proof: Let $\overline{A} \subseteq U$ where $U$ is $s$-open set, since $\overline{\overline{A}} = \overline{A} = A$ [Proposition(2.11)(9)]. Then $\overline{\overline{A}} = A \Rightarrow \overline{\overline{A}} \subseteq U$ where $U$ is $s$-open $\Rightarrow \overline{A}$ is $sf$-open set.

**Proposition(3.14):** Assume that $X$ is an $tp\cdot s$ & $A \subseteq X$, if $A$ is $s$-closed and pre closed then $A$ is $sf$-open set.

Proof: Let $A$ is $s$-closed then $A^{s} = \overline{A}$, since $A$ pre closed then $\overline{A} \subseteq A$, but $A^{s} = \overline{A}$ then $\overline{\overline{A}} \subseteq A \Rightarrow A$ $f$-closed by using [Remark(3.3)] $A$ is $sf$-open set.

**Definition(3.15):** Assume that $X$ is a $tp\cdot s$ & $A \subseteq X$. Then the intersection of all $sf$-closed of $X$ which containing $A$ is named $sf$-closure of $A$ and shortened by $\overline{A}^{sf}$, that means $\overline{A}^{sf} = \cap\{F:F$ is $sf$-closed in $X\}$.

**Lemma(3.16):** Assume that $X$ is a $tp\cdot s$ & $A \subseteq X$. Then $x \in \overline{A}^{sf}$ iff for all $sf$-open set $G$ and $x \in G \ , \ G \cap A \neq \emptyset$.

Proof: (⇒) Assume that $x \notin \overline{A}$ then $x \notin \cap\{F:F$ is $sf$-closed in $X\}$ and $A \subseteq F$, then $x \in [\cap F]^{c}$, $[\cap F]^{c}$ $sf$-open containing $x$. Hence $[\cap F]^{c} \cap A \subseteq [\cap F]^{c} \cap [\cap F] = \emptyset$. (⇐) Conversely, Suppose that $\exists$ $sf$-open set $G$ so that $x \in G$, $G \cap A = \emptyset$ then $A \subseteq G^{c}$, $G^{c}$ is $sf$-closed hence $x \notin \overline{A}^{sf}$.

**Definition(3.17):** Let $X$ be a $tp\cdot s\ , \ x \in X$ & $A \subseteq X$. The point $x$ is called $sf$-limit point of $A$ if each $sf$-open set containing $U$, contains a point of $A$ distinct from $x$. We shall call the set of all $sf$-limit point of $A$ the $sf$- derivative set of $A$ and denoted by $A^{isf}$. Therefore $x \in A^{isf}$ if for every $sf$-open set $U$ in $X$ such $x \in V$ implies that $\cap(A - \{x\}) = \emptyset$.

**Proposition(3.18):** Let $X$ be a $tp\cdot s$ and $A \subseteq B \subseteq X$. Then:

1. $\overline{A}^{sf} = A \cup A^{isf}$.
2. $A$ is an $sf$-closed set iff $A^{isf} \subseteq A$.
3. $A^{isf} \subseteq B^{isf}$.

Proof: 1- By definition $A \subseteq \overline{A}^{sf}$ ..... (1). Let $x \in A^{isf} \Rightarrow x \notin A$. Then $\forall$ $sf$-open set $U$ contained $x$, then $(U \cap A) - \{x\} = \emptyset$. Then $\forall sf$-open set in $U$ contained $x$, then $U \cap A = \emptyset$ by [Lemma(3.16)]. Then $x \in \overline{A}^{sf} \Rightarrow A^{isf} \subseteq \overline{A}^{sf}$ ..... (2). From (1) and (2) $A \cup A^{isf} \subseteq \overline{A}^{sf}$.

Let $x \in \overline{A}^{sf}$. Since $A \subseteq \overline{A}^{sf}$ by definition and $\forall x \in A^{sf}$ then either $x \in A$ or $x \notin A$. If $x \in A$ then $(U \cap A) - \{x\} \neq \emptyset$. Since $x \in \overline{A}^{sf} \Rightarrow \forall$ sf-open set $U$ contained $x$, then $U \cap A = \emptyset$, since $x \notin A$ then $(U \cap A) - \{x\} = \emptyset$. Then $x \in A^{isf} \Rightarrow x \in A \cup A^{isf}$ then $\overline{A}^{sf} \subseteq A \cup A^{isf}$ then $A^{isf} = A \cup A^{isf}$.
2- \(\Rightarrow\) Let \(A^{sf} \subseteq A\). \(\overline{A}^{sf} = A \cup A^{sf} \subseteq A\), since \(A \subseteq \overline{A}^{sf}\) then \(A = \overline{A}^{sf}\), then \(A\) is an \(sf\)-closed set.

\(\Leftarrow\) Let \(A\) be \(sf\)-closed set. Thus \(A = \overline{A}^{sf}\) from [proposition (3.18)(1)]. \(A = A \cup A^{sf}\) then \(A^{sf} \subseteq A\).

3- Let \(A \subseteq B\) and let \(x \in A^{sf}, \forall U\) is \(sf\)-open set contained \(x\) then \((U \cap A) - \{x\} \neq \emptyset\). Since \(A \subseteq B\) \(\Rightarrow (U \cap B) - \{x\} \neq \emptyset\). Then \(x \in B^{sf}\) then \(A^{sf} \subseteq B^{sf}\).

**Remark (3.19):** Assume that \(X\) is a \(tp\)-\(s\) \& \(A \subseteq X\), then \(\overline{A}^{sf}\) is smallest \(sf\)-closed set containing \(A\).

proof: Suppose that \(B\) is \(sf\)-closed set contend such that \(A \subseteq B\) since \(\overline{A}^{sf} = A \cup A^{sf}\). And \(\overline{A}^{sf} \subseteq \overline{B}^{sf}, A \subseteq B\), then \(\overline{A}^{sf} = A \cup A^{sf} \subseteq A \cup \overline{A}^{sf} \subseteq B\), then \(\overline{A}^{sf} \subseteq B\) therefore \(\overline{A}^{sf}\) is smallest \(sf\)-closed set contained \(A\).

**Proposition (3.20):** Let \(X\) be a \(tp\)-\(s\) \& \(A, B\) are subset of \(X\) with \(B\) \(sf\)-open set. If \(x \in B\) and \(B \cap A = \emptyset\) then \(x \notin \overline{A}^{sf}\).

proof: Suppose \(x \in \overline{A}^{sf}\), then either \(x \in A\) or \(x \in A^{sf}\). If \(x \in A\), then \(B \cap A \neq \emptyset\) which contradicts the assumption and if \(x \in A^{sf}\) and \(x \notin A\), then \((B \cap A) - \{x\} \neq \emptyset\) for every \(sf\)-open \(G\) in \(X\) containing \(x\) and hence \(G \cap A \neq \emptyset\) which is a contradiction since \(B\) is \(sf\)-open set containing \(x\) and \(B \cap A = \emptyset\) and hence \(x \notin \overline{A}^{sf}\).

**Definition (3.21):** Assume that \(X\) is a \(tp\)-\(s\) \& \(B \subseteq X\). An \(sf\)-neighborhood of \(B\) is any subset of \(X\) which contains an \(sf\)-open set containing \(B\). The \(sf\)-neighborhood of a subset \(\{x\}\) is also called \(sf\)-neighborhood of the point \(x\).

**Definition (3.22):** Assume that \(A\) is a subset of \(a\) \(tp\)-\(s\) \(X\). For each \(x \in X\), then \(x\) is said to be \(sf\)-boundary point of \(A\) if each \(sf\)-neighborhood \(U_x\) of \(x\), we have \(U_x \cap A \neq \emptyset\) and \(U_x \cap A^c \neq \emptyset\). The set of all \(sf\)-boundary point of \(A\) is denoted by \(b_{sf}(A)\).

**Proposition (3.23):** Assume that \(X\) is a \(tp\)-\(s\) and \(A, B \subseteq X\), then

1. \(A\) is an \(sf\)-closed set \(\iff A = \overline{A}^{sf}\).
2. \(\overline{A}^{sf} \subseteq \overline{A}\).
3. \(\overline{A}^{sf} = \overline{A}^{sf} \overline{A}^{sf}\).
4. If \(A \subseteq B\) then \(\overline{A}^{sf} \subseteq \overline{B}^{sf}\).

proof: 1- \(\Rightarrow\) Let \(A\) is an \(sf\)-closed set. Since \(A \subseteq \overline{A}^{sf}\). Then \(\overline{A}^{sf} \subseteq A\) (since \(\overline{A}^{sf}\) is the smallest \(sf\)-closed set containing \(A\)), then \(A = \overline{A}^{sf}\).
(⇐) Let \( A^{sf} = A \). Then \( A^{sf} \) is an \( sf \)-closed set as \( A = A^{sf} \Rightarrow A \) is a \( sf \)-closed set.

2- Let \( x \in \overline{A}^{sf} \) and \( A \) is a \( sf \)-closed set, then \( A = \overline{A}^{sf} \Rightarrow x \in A \subseteq A \). Then \( x \in \overline{A} \). Therefore \( \overline{A}^{sf} \subseteq \overline{A} \).

3- Since \( \overline{A} \) is \( sf \)-closed set, then \( \overline{A} = \overline{A^{sf}} \) by (2).

4- Let \( A \subseteq B \) and \( B \subseteq \overline{B}^{sf} \), then \( A \subseteq \overline{B}^{sf} \Rightarrow \overline{B}^{sf} \) is a \( sf \)-closed set containing \( A \). Since \( \overline{A}^{sf} \) is smallest \( sf \)-closed set containing \( A \). Then \( \overline{A}^{sf} \subseteq \overline{B}^{sf} \).

**Definition (3.24):** Assume that \( X \) is tp-\( s \) and \( A \subseteq X \). The union of all \( sf \)-open sets of \( X \) contained in \( A \) is named \( sf \)-Interior of \( A \), shortened by \( A^{c}sf \) or \( sf-\text{In}_r(A) \), that means \( sf-\text{In}_r(A) = \bigcup \{B:B \) is \( sf \)-open in \( X \) and \( B \subseteq A \} \).

**Proposition (3.25):** Assume that \( X \) is tp-\( s \) and \( A \subseteq X \). Then \( \overline{A}^{sf} = \left( A^{c}sf \right)^{c} \).

Proof: Since \( A \subseteq \overline{A}^{sf} \Rightarrow \overline{A}^{sf} \subseteq A^{c} \Rightarrow \overline{A}^{csf^{c}} \subseteq \overline{A}^{sf} \subseteq A^{c}sf \Rightarrow \overline{A}^{sf} \subseteq A^{c}sf \Rightarrow A^{c}sf \subseteq \overline{A}^{sf} \subseteq \overline{A}^{sf} \) ......(1). Since \( A^{c}sf \subseteq A^{c} \Rightarrow A \subseteq A^{c}sf \Rightarrow A \subseteq A^{c}sf \Rightarrow A^{c}sf \subseteq \overline{A}^{sf} \subseteq A^{c}sf \overline{A}^{sf} \subseteq A^{c}sf \) ......(2). From (1) and (2) we get \( \overline{A}^{sf} = \left( A^{c}sf \right)^{c} \).

**Proposition (3.26):** Assume that \( X \) is tp-\( s \) and \( A \subseteq X \). Then \( x \in A^{sf} \iff \) there is an \( sf \)-open set \( U \) containing \( x \) so that \( x \in U \subseteq A \).

Proof: Assume that \( x \in A^{sf} \iff x \in \bigcup \{U:U \subseteq A \) such that \( U \) is \( sf \)-open in \( X \} \iff \exists U \) is \( sf \)-open in \( X \) so that \( x \in U \subseteq A \).

**Proposition (3.27):** Assume that \( X \) is tp-\( s \) and \( A \subseteq B \subseteq X \), then:

1. \( A^{sf} \) is an \( sf \)-open set.
2. \( A \) is an \( sf \)-open set iff \( A = A^{sf} \).
3. \( A^{sf} = A^{sf}^{csf} \).
4. If \( A \subseteq B \) then \( A^{sf} \subseteq B^{sf} \).

Proof: 1- \( A^{sf} = \bigcup \{B:B \) is \( sf \)-open and \( B \subseteq A \} \), by [proposition (3.7)]. Then \( A^{sf} \) is an \( sf \)-open set.

2- (⇒) Let \( A \) be an \( sf \)-open set from definition \( A^{sf} \subseteq A \), \( A^{sf} = \bigcup \{U:U \subseteq A \) \( U \) is an \( sf \)-open set in \( X \} \). Since \( A \) is \( sf \)-open set in \( X \). Then \( A \subseteq A^{sf} \Rightarrow A = A^{sf} \).

(⇐) Let \( A = A^{sf} \), since \( A^{sf} \) is the union \( sf \)-open sets and since \( A^{sf} = A \Rightarrow A \) is a \( sf \)-open set.
3- Let $A^{sf} = \cup \{B: B$ is an $sf$-open set in $X$ and $B \subseteq A\}$. Then $A^{sf} = A^{sf} \cap A$. By (2) $A = A^{sf}$. Then $A^{sf} = A^{sf} \cap A$.

4- Let $A \subseteq B$ & $x \in A^{sf}$. Then $\exists A$-open $U$ in $X$ such that $x \in U \subseteq A$. Since $A \subseteq B$. Then $\exists A$-open $U$ in $X$ such that $x \in U \subseteq A \subseteq B$. Then $A^{sf} \subseteq B^{sf}$.

**Proposition (3.28):** Assume that $X$ is a $tp$-$s$ & $A \subseteq X$. Then:

1. $b_{sf}(A) = A^{sf} \cap A^{cf}$.
2. $A^{sf} = A - b_{sf}(A)$.
3. $A^{sf} = A \cup b_{sf}(A)$.

Proof: Clear

**Proposition (3.29):** Assume that $X$ is a $tp$-$s$ & $A \subseteq X$. Then:

1. $A^{sf} = A^{sf} \cup b_{sf}(A)$.
2. $A$ is an $sf$-open set $\iff b_{sf}(A) \subseteq A^{c}$.
3. $(A^{sf})^{c} = (A^{cf})^{c}$.

Proof: Clear

**References**


